1.4 Solving Absolute Value Equations

Essential Question How can you solve an absolute value equation?

EXPLORATION 1 Solving an Absolute Value Equation Algebraically

Work with a partner. Consider the absolute value equation

$$|x + 2| = 3.$$

a. Describe the values of $x + 2$ that make the equation true. Use your description to write two linear equations that represent the solutions of the absolute value equation.

b. Use the linear equations you wrote in part (a) to find the solutions of the absolute value equation.

c. How can you use linear equations to solve an absolute value equation?

MAKING SENSE OF PROBLEMS

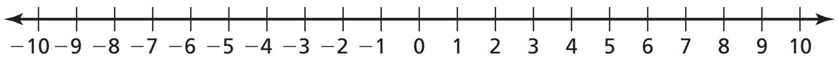
To be proficient in math, you need to explain to yourself the meaning of a problem and look for entry points to its solution.

EXPLORATION 2 Solving an Absolute Value Equation Graphically

Work with a partner. Consider the absolute value equation

$$|x + 2| = 3.$$

a. On a real number line, locate the point for which $x + 2 = 0$.



b. Locate the points that are 3 units from the point you found in part (a). What do you notice about these points?

c. How can you use a number line to solve an absolute value equation?

EXPLORATION 3 Solving an Absolute Value Equation Numerically

Work with a partner. Consider the absolute value equation

$$|x + 2| = 3.$$

a. Use a spreadsheet, as shown, to solve the absolute value equation.

b. Compare the solutions you found using the spreadsheet with those you found in Explorations 1 and 2. What do you notice?

c. How can you use a spreadsheet to solve an absolute value equation?

	A	B
1	*x*	|*x* + 2|
2	-6	4
3	-5	
4	-4	
5	-3	
6	-2	
7	-1	
8	0	
9	1	
10	2	
11		

abs(A2 + 2)

Communicate Your Answer

4. How can you solve an absolute value equation?

5. What do you like or dislike about the algebraic, graphical, and numerical methods for solving an absolute value equation? Give reasons for your answers.

1.4 Lesson

Core Vocabulary

absolute value equation, *p. 28*
extraneous solution, *p. 31*

Previous
absolute value
opposite

What You Will Learn

- Solve absolute value equations.
- Solve equations involving two absolute values.
- Identify special solutions of absolute value equations.

Solving Absolute Value Equations

An **absolute value equation** is an equation that contains an absolute value expression. You can solve these types of equations by solving two related linear equations.

Core Concept

Properties of Absolute Value

Let a and b be real numbers. Then the following properties are true.

1. $|a| \geq 0$

2. $|-a| = |a|$

3. $|ab| = |a||b|$

4. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$

Solving Absolute Value Equations

To solve $|ax + b| = c$ when $c \geq 0$, solve the related linear equations

$$ax + b = c \quad \text{or} \quad ax + b = -c.$$

When $c < 0$, the absolute value equation $|ax + b| = c$ has no solution because absolute value always indicates a number that is not negative.

EXAMPLE 1 Solving Absolute Value Equations

Solve each equation. Graph the solutions, if possible.

a. $|x - 4| = 6$

b. $|3x + 1| = -5$

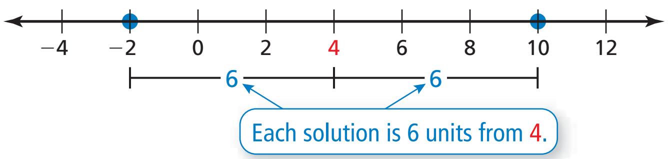
SOLUTION

a. Write the two related linear equations for $|x - 4| = 6$. Then solve.

$x - 4 = 6 \quad \text{or} \quad x - 4 = -6$ Write related linear equations.

$x = 10 \quad\quad\quad x = -2$ Add 4 to each side.

▶ The solutions are $x = 10$ and $x = -2$.



Property of Absolute Value

b. The absolute value of an expression must be greater than or equal to 0. The expression $|3x + 1|$ cannot equal -5.

▶ So, the equation has no solution.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Solve the equation. Graph the solutions, if possible.

1. $|x| = 10$

2. $|x - 1| = 4$

3. $|3 + x| = -3$

EXAMPLE 2 Solving an Absolute Value Equation

Solve $|3x + 9| - 10 = -4$.

SOLUTION

First isolate the absolute value expression on one side of the equation.

$|3x + 9| - 10 = -4$ Write the equation.

$|3x + 9| = 6$ Add 10 to each side.

Now write two related linear equations for $|3x + 9| = 6$. Then solve.

$3x + 9 = 6$ *or* $3x + 9 = -6$ Write related linear equations.

$3x = -3$ $3x = -15$ Subtract 9 from each side.

$x = -1$ $x = -5$ Divide each side by 3.

▶ The solutions are $x = -1$ and $x = -5$.

ANOTHER WAY

Using the product property of absolute value, $|ab| = |a|\,|b|$, you could rewrite the equation as

$$3|x + 3| - 10 = -4$$

and then solve.

EXAMPLE 3 Writing an Absolute Value Equation

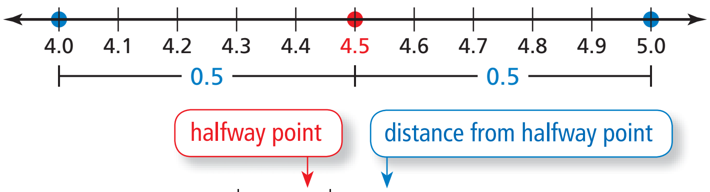
In a cheerleading competition, the minimum length of a routine is 4 minutes. The maximum length of a routine is 5 minutes. Write an absolute value equation that represents the minimum and maximum lengths.

REASONING

Mathematically proficient students have the ability to decontextualize problem situations.

SOLUTION

1. **Understand the Problem** You know the minimum and maximum lengths. You are asked to write an absolute value equation that represents these lengths.
2. **Make a Plan** Consider the minimum and maximum lengths as solutions to an absolute value equation. Use a number line to find the halfway point between the solutions. Then use the halfway point and the distance to each solution to write an absolute value equation.
3. **Solve the Problem**



$$|x - 4.5| = 0.5$$

▶ The equation is $|x - 4.5| = 0.5$.

4. **Look Back** To check that your equation is reasonable, substitute the minimum and maximum lengths into the equation and simplify.

Minimum	**Maximum**
$|4 - 4.5| = 0.5$ ✓	$|5 - 4.5| = 0.5$ ✓

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Solve the equation. Check your solutions.

4. $|x - 2| + 5 = 9$

5. $4|2x + 7| = 16$

6. $-2|5x - 1| - 3 = -11$

7. For a poetry contest, the minimum length of a poem is 16 lines. The maximum length is 32 lines. Write an absolute value equation that represents the minimum and maximum lengths.

Solving Equations with Two Absolute Values

If the absolute values of two algebraic expressions are equal, then they must either be equal to each other or be opposites of each other.

Core Concept

Solving Equations with Two Absolute Values

To solve $|ax + b| = |cx + d|$, solve the related linear equations

$$ax + b = cx + d \quad \text{or} \quad ax + b = -(cx + d).$$

EXAMPLE 4 Solving Equations with Two Absolute Values

Solve (a) $|3x - 4| = |x|$ and (b) $|4x - 10| = 2|3x + 1|$.

SOLUTION

a. Write the two related linear equations for $|3x - 4| = |x|$. Then solve.

$$\begin{aligned} 3x - 4 &= x \\ -x \quad & \quad -x \\ 2x - 4 &= 0 \\ +4 \quad & \quad +4 \\ 2x &= 4 \\ \frac{2x}{2} &= \frac{4}{2} \\ x &= 2 \end{aligned} \qquad or \qquad \begin{aligned} 3x - 4 &= -x \\ +x \quad & \quad +x \\ 4x - 4 &= 0 \\ +4 \quad & \quad +4 \\ 4x &= 4 \\ \frac{4x}{4} &= \frac{4}{4} \\ x &= 1 \end{aligned}$$

▶ The solutions are $x = 2$ and $x = 1$.

Check

$$\begin{aligned} |3x - 4| &= |x| \\ |3(2) - 4| &\stackrel{?}{=} |2| \\ |2| &\stackrel{?}{=} |2| \\ 2 &= 2 \; ✓ \end{aligned}$$

$$\begin{aligned} |3x - 4| &= |x| \\ |3(1) - 4| &\stackrel{?}{=} |1| \\ |-1| &\stackrel{?}{=} |1| \\ 1 &= 1 \; ✓ \end{aligned}$$

b. Write the two related linear equations for $|4x - 10| = 2|3x + 1|$. Then solve.

$$\begin{aligned} 4x - 10 &= 2(3x + 1) \\ 4x - 10 &= 6x + 2 \\ -6x \quad & \quad -6x \\ -2x - 10 &= 2 \\ +10 \quad & \quad +10 \\ -2x &= 12 \\ \frac{-2x}{-2} &= \frac{12}{-2} \\ x &= -6 \end{aligned} \qquad or \qquad \begin{aligned} 4x - 10 &= 2[-(3x + 1)] \\ 4x - 10 &= 2(-3x - 1) \\ 4x - 10 &= -6x - 2 \\ +6x \quad & \quad +6x \\ 10x - 10 &= -2 \\ +10 \quad & \quad +10 \\ 10x &= 8 \\ \frac{10x}{10} &= \frac{8}{10} \\ x &= 0.8 \end{aligned}$$

▶ The solutions are $x = -6$ and $x = 0.8$.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Solve the equation. Check your solutions.

8. $|x + 8| = |2x + 1|$

9. $3|x - 4| = |2x + 5|$

Identifying Special Solutions

When you solve an absolute value equation, it is possible for a solution to be *extraneous*. An **extraneous solution** is an apparent solution that must be rejected because it does not satisfy the original equation.

EXAMPLE 5 Identifying Extraneous Solutions

Solve $|2x + 12| = 4x$. Check your solutions.

SOLUTION

Write the two related linear equations for $|2x + 12| = 4x$. Then solve.

$2x + 12 = 4x$	*or*	$2x + 12 = -4x$	Write related linear equations.
$12 = 2x$		$12 = -6x$	Subtract $2x$ from each side.
$6 = x$		$-2 = x$	Solve for x.

Check the apparent solutions to see if either is extraneous.

Check

$|2x + 12| = 4x$

$|2(6) + 12| \stackrel{?}{=} 4(6)$

$|24| \stackrel{?}{=} 24$

$24 = 24$ ✓

$|2x + 12| = 4x$

$|2(-2) + 12| \stackrel{?}{=} 4(-2)$

$|8| \stackrel{?}{=} -8$

$8 \neq -8$ ✗

▶ The solution is $x = 6$. Reject $x = -2$ because it is extraneous.

When solving equations of the form $|ax + b| = |cx + d|$, it is possible that one of the related linear equations will not have a solution.

EXAMPLE 6 Solving an Equation with Two Absolute Values

Solve $|x + 5| = |x + 11|$.

SOLUTION

By equating the expression $x + 5$ and the opposite of $x + 11$, you obtain

$x + 5 = -(x + 11)$	Write related linear equation.
$x + 5 = -x - 11$	Distributive Property
$2x + 5 = -11$	Add x to each side.
$2x = -16$	Subtract 5 from each side.
$x = -8.$	Divide each side by 2.

However, by equating the expressions $x + 5$ and $x + 11$, you obtain

$x + 5 = x + 11$	Write related linear equation.
$x = x + 6$	Subtract 5 from each side.
$0 = 6$ ✗	Subtract x from each side.

which is a false statement. So, the original equation has only one solution.

REMEMBER

Always check your solutions in the original equation to make sure they are not extraneous.

▶ The solution is $x = -8$.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Solve the equation. Check your solutions.

10. $|x + 6| = 2x$

11. $|3x - 2| = x$

12. $|2 + x| = |x - 8|$

13. $|5x - 2| = |5x + 4|$

1.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** What is an extraneous solution?

2. **WRITING** Without calculating, how do you know that the equation $|4x - 7| = -1$ has no solution?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, simplify the expression.

3. $|-9|$
4. $-|15|$
5. $|14| - |-14|$
6. $|-3| + |3|$
7. $-|-5 \cdot (-7)|$
8. $|-0.8 \cdot 10|$
9. $\left|\frac{27}{-3}\right|$
10. $\left|-\frac{-12}{4}\right|$

In Exercises 11–24, solve the equation. Graph the solution(s), if possible. *(See Examples 1 and 2.)*

11. $|w| = 6$
12. $|r| = -2$
13. $|y| = -18$
14. $|x| = 13$
15. $|m + 3| = 7$
16. $|q - 8| = 14$
17. $|-3d| = 15$
18. $\left|\frac{t}{2}\right| = 6$
19. $|4b - 5| = 19$
20. $|x - 1| + 5 = 2$
21. $-4|8 - 5n| = 13$
22. $-3\left|1 - \frac{2}{3}v\right| = -9$
23. $3 = -2\left|\frac{1}{4}s - 5\right| + 3$
24. $9|4p + 2| + 8 = 35$

25. **WRITING EQUATIONS** The minimum distance from Earth to the Sun is 91.4 million miles. The maximum distance is 94.5 million miles. *(See Example 3.)*

    a. Represent these two distances on a number line.

    b. Write an absolute value equation that represents the minimum and maximum distances.

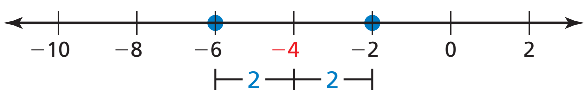
26. **WRITING EQUATIONS** The shoulder heights of the shortest and tallest miniature poodles are shown.

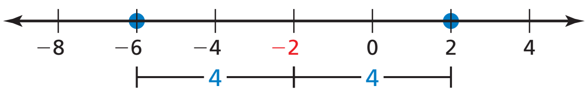
    a. Represent these two heights on a number line.

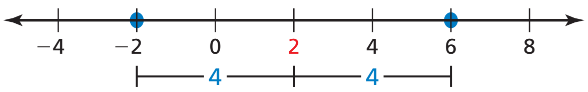
    b. Write an absolute value equation that represents these heights.

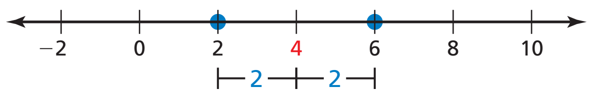
USING STRUCTURE In Exercises 27–30, match the absolute value equation with its graph without solving the equation.

27. $|x + 2| = 4$
28. $|x - 4| = 2$
29. $|x - 2| = 4$
30. $|x + 4| = 2$

A.


B.


C.


D.


In Exercises 31–34, write an absolute value equation that has the given solutions.

31. $x = 8$ and $x = 18$

32. $x = -6$ and $x = 10$

33. $x = 2$ and $x = 9$

34. $x = -10$ and $x = -5$

In Exercises 35–44, solve the equation. Check your solutions. *(See Examples 4, 5, and 6.)*

35. $|4n - 15| = |n|$

36. $|2c + 8| = |10c|$

37. $|2b - 9| = |b - 6|$

38. $|3k - 2| = 2|k + 2|$

39. $4|p - 3| = |2p + 8|$

40. $2|4w - 1| = 3|4w + 2|$

41. $|3h + 1| = 7h$

42. $|6a - 5| = 4a$

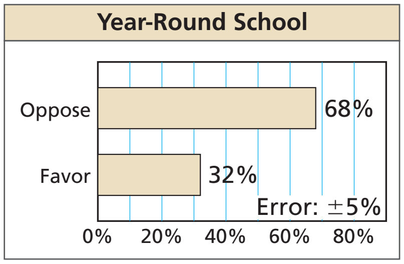
43. $|f - 6| = |f + 8|$

44. $|3x - 4| = |3x - 5|$

45. MODELING WITH MATHEMATICS Starting from 300 feet away, a car drives toward you. It then passes by you at a speed of 48 feet per second. The distance d (in feet) of the car from you after t seconds is given by the equation $d = |300 - 48t|$. At what times is the car 60 feet from you?

46. MAKING AN ARGUMENT Your friend says that the absolute value equation $|3x + 8| - 9 = -5$ has no solution because the constant on the right side of the equation is negative. Is your friend correct? Explain.

47. MODELING WITH MATHEMATICS You randomly survey students about year-round school. The results are shown in the graph.



The error given in the graph means that the actual percent could be 5% more or 5% less than the percent reported by the survey.

a. Write and solve an absolute value equation to find the least and greatest percents of students who could be in favor of year-round school.

b. A classmate claims that $\frac{1}{3}$ of the student body is actually in favor of year-round school. Does this conflict with the survey data? Explain.

48. MODELING WITH MATHEMATICS The recommended weight of a soccer ball is 430 grams. The actual weight is allowed to vary by up to 20 grams.

a. Write and solve an absolute value equation to find the minimum and maximum acceptable soccer ball weights.

b. A soccer ball weighs 423 grams. Due to wear and tear, the weight of the ball decreases by 16 grams. Is the weight acceptable? Explain.

ERROR ANALYSIS In Exercises 49 and 50, describe and correct the error in solving the equation.

49.

✗

$$|2x - 1| = -9$$

$2x - 1 = -9$ *or* $2x - 1 = -(-9)$

$2x = -8$ $\quad$ $2x = 10$

$x = -4$ $\quad$ $x = 5$

The solutions are $x = -4$ and $x = 5$.

50.

✗

$$|5x + 8| = x$$

$5x + 8 = x$ *or* $5x + 8 = -x$

$4x + 8 = 0$ $\quad$ $6x + 8 = 0$

$4x = -8$ $\quad$ $6x = -8$

$x = -2$ $\quad$ $x = -\frac{4}{3}$

The solutions are $x = -2$ and $x = -\frac{4}{3}$.

51. ANALYZING EQUATIONS Without solving completely, place each equation into one of the three categories.

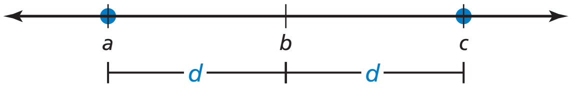
No solution	One solution	Two solutions

$|x - 2| + 6 = 0$ $\quad$ $|x + 3| - 1 = 0$

$|x + 8| + 2 = 7$ $\quad$ $|x - 1| + 4 = 4$

$|x - 6| - 5 = -9$ $\quad$ $|x + 5| - 8 = -8$

52. **USING STRUCTURE** Fill in the equation $|x - \square| = \square$ with *a*, *b*, *c*, or *d* so that the equation is graphed correctly.



ABSTRACT REASONING In Exercises 53–56, complete the statement with *always*, *sometimes*, or *never*. Explain your reasoning.

53. If $x^2 = a^2$, then $|x|$ is ________ equal to $|a|$.

54. If *a* and *b* are real numbers, then $|a - b|$ is ________ equal to $|b - a|$.

55. For any real number *p*, the equation $|x - 4| = p$ will ________ have two solutions.

56. For any real number *p*, the equation $|x - p| = 4$ will ________ have two solutions.

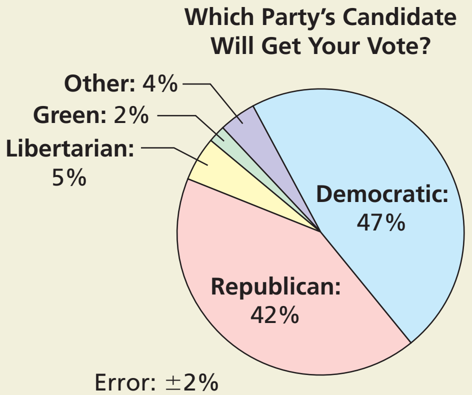
57. **WRITING** Explain why absolute value equations can have no solution, one solution, or two solutions. Give an example of each case.

58. **THOUGHT PROVOKING** Describe a real-life situation that can be modeled by an absolute value equation with the solutions $x = 62$ and $x = 72$.

59. **CRITICAL THINKING** Solve the equation shown. Explain how you found your solution(s).

$$8|x + 2| - 6 = 5|x + 2| + 3$$

60. **HOW DO YOU SEE IT?** The circle graph shows the results of a survey of registered voters the day of an election.



The error given in the graph means that the actual percent could be 2% more or 2% less than the percent reported by the survey.

a. What are the minimum and maximum percents of voters who could vote Republican? Green?

b. How can you use absolute value equations to represent your answers in part (a)?

c. One candidate receives 44% of the vote. Which party does the candidate belong to? Explain.

61. **ABSTRACT REASONING** How many solutions does the equation $a|x + b| + c = d$ have when $a > 0$ and $c = d$? when $a < 0$ and $c > d$? Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Identify the property of equality that makes Equation 1 and Equation 2 equivalent. *(Section 1.1)*

62.

Equation 1	$3x + 8 = x - 1$
Equation 2	$3x + 9 = x$

63.

Equation 1	$4y = 28$
Equation 2	$y = 7$

Use a geometric formula to solve the problem. *(Skills Review Handbook)*

64. A square has an area of 81 square meters. Find the side length.

65. A circle has an area of 36π square inches. Find the radius.

66. A triangle has a height of 8 feet and an area of 48 square feet. Find the base.

67. A rectangle has a width of 4 centimeters and a perimeter of 26 centimeters. Find the length.